Indian Statistical Institute, Bangalore Centre.
Mid-Semester Exam : Probability 1

Instructor : Yogeshwaran D.

Date : September 12th , 2019.

Max. points : 10. **Time Limit : 1.5 hours.**

Answer any two questions only. All questions carry 5 points.

Give necessary justifications and explanations for all your arguments. If you are citing results from the class or assignments, mention it clearly.
Always define the underlying probability spaces, events and random variables clearly before computing anything !

1. A drunkard has n keys in his pocket and only one of the keys is the correct one to open his house. For two successive nights, he has forgotten the correct key to open his house. Compute the expected number of attempts in the following two scenarios :

   - Sampling without replacement:
     On the first night, he tries the keys from his pocket at random one by one but remembers not to try the same key twice. Let N stand for the random variable denoting the number of attempts (including the successful one)to open the door. Compute $\mathbb{E}(N)$.
   - Sampling with replacement:
     On the second night, he still tries keys from his pocket at random but he is so drunk that he puts back the unsuccessful keys again in his pocket. So, it is possible that he can select the same key multiple times. If he does not succeed in n attempts, he gives up. Here, let N stand for the random variable denoting the number of unsuccessful attempts (in the case he has given up, the number of unsuccessful attempts is to be taken n). Compute $\mathbb{E}(N)$.

2. A box contains n coupons labelled $1, 2, \ldots, n$. Coupons are drawn one after another without replacement (a coupon is drawn, the number

noted and discarded, the next coupon is drawn, etc). In each of the following cases, prove or disprove that A and B are independent and if they are not independent compute $\mathbb{P}(B|A)$ and $\mathbb{P}(B)$.

(a) Let A be the event that the first coupon drawn is an even number. Let B be the event that the second coupon drawn is an even number.

(b) Let A be the event that the first coupon drawn is an even number. Let B be the event that the second coupon drawn is an odd number.

(c) Let A be the event that the first coupon draw is at most 4 and B be the event that the second coupon drawn is at least 7.

3. Fix $n \geq 1$ and suppose that n is divisible by $k_1, \ldots, k_m$ where $k_1, \ldots, k_m$ are distinct natural numbers. We pick a random number j uniformly from $[n]$. Let A_k the event that j is divisible by k.

   (a) Compute $\mathbb{P}(A_{k_1} \cap A_{k_2} \cap \ldots \cap A_{k_n})$ for any $k_1, k_2, \ldots, k_m$ (not necessarily co-prime).

   (b) Let $k_1, \ldots, k_3$ be pairwise coprime ($gcd(k_1, k_2), gcd(k_2, k_3), gcd(k_1, k_3) =$ 1). Prove that $A_{k_1}, \ldots, A_{k_3}$ are independent.

   (c) If $k_1, \ldots, k_3$ are coprime (i.e., $gcd(k_1, k_2, k_3) = 1$) but not pairwise comprime. Are $A_{k_1}, A_{k_2}, A_{k_3}$ independent ?

4. (a) A sample of size r is taken from a set of n items with replacement and uniformly at random. Find the probability that N given items (say $1, \ldots, N$) will all be included in the sample.

   (b) A sample of size r is taken from a set of n items without replacement and uniformly at random. Find the probability that N given items (say $1, \ldots, N$) will all be included in the sample.